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Term Premium
Evolution for Germany and Portugal

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Abstract

In this study, we implement a particular model in the class of the Affine Arbitrage-Free Nelson Siegel Term Structure models [1] and use it to obtain the Term Premium – the difference between long-term yields and the expected value of short term yields. To calibrate the model the linear Kalman filter [6] was considered using zero coupon rates constructed from government yields observed in the market. Results are obtained for Germany and Portugal. The results can be summarized as follows: for Germany, the term premium evolution obtained indicates that we can presently 'observe' the term premium near a historical minimum, and, moreover, it is negative. Adding to this, the fact that rates are also at a minimum and that the model predicts a term premium increase, a discretionary issuer would certainly choose longer maturities. For Portugal, in spite of the fact that long-term yields are also below the historical average – which would indicate a duration extension – the model returns a relatively high term premium compared to what was historically observed, but close to what the model predicts is the steady state, which means that - considering the low interest rate environment – a discretionary decision maker would also consider a maturity extension. Any conclusion based on the term premium for Portugal needs to be viewed with care, because it can be related to the credit risk level (as shown in section 3.2), and therefore the steady state (based on historical data) can be 'contaminated' by the period of the financial assistance program. Finally, we found a method to explain the term premium from variables observed in the market. The German term premium shows strong correlation with inflation expectations, whereas the term premium of Portugal follows much more closely the general credit risk perceived in the market

Key words: Term Premium, Affine Arbitrage-Free Nelson-Siegel, Kalman filter, Maximum Likelihood

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1 Introduction

Our objective is to illustrate the implementation of a particular model in the class of the Affine Arbitrage-Free Nelson-Siegel Term Structure models [1] and use it to estimate the Term Premium – the difference between long-term yields and the expected value of short-term yields.

The estimated Term Premia can be interpreted as the price of risk: how much we need to pay over interest rate expectations to issue long bonds instead of short ones. We can compare the current values for the Term Premium both with the historical values and forecasts of that measure. From these comparisons one can answer questions like "what to do with the portfolio duration".

The model is estimated using historical values of Portuguese and German bonds and results are analysed. Finally, we try to find the determinants of the term premium for both countries.

2 Modelling the Term Premium

The Term Premium is the price of risk. We can think of it as the cost that the issuer bears to reduce refinancing risk, or as the value charged by buyers to bear interest rate risk. It is the annualized rate difference from issuing a m-year bond and rolling-over m/n times a n-year bond.

$$TP_{(n)}^{(m)} = y_t^{(m)} - \frac{n}{m}\sum_{i=1}^{\frac{m}{n}} E_{\mathbb{P}}\left[y_{t+(i-1)n}^{(n)} \middle| \Im_t\right] \tag{1}$$

Expectations are obtained in the physical measure $\mathbb{P}$. If instead the risk-neutral measure $\mathbb{Q}$ was used, one would get $TP_{(n)}^{(m)} = 0$.

From Equation (1) we can see that to obtain the term premium we need to have a model to forecast interest rates in the physical measure.

2.1 The Arbitrage Free Nelson-Siegel Model

The model that we have selected for this job is a special case of the Affine Arbitrage-Free Class of Nelson-Siegel Term Structure models (AFNS) [1].

These models make use of latent factors X_t, essentially affine risk premia [2] and the framework of Affine Term Structure models [3] to obtain an equation for yields as a function of the factors that resemble those of Nelson-Siegel [4].

With this specification we obtain a model that is arbitrage-free – a feature which is not present at Nelson-Siegel (nor its dynamic version of Diebold and Li [5]) –, but with an equation for rates where the loadings are the ones of Nelson-Siegel. Because the factors of Nelson-Siegel are identified as level, slope and curvature, we have a simple interpretation for them, and also an easier numerical estimation compared with the canonical affine models [3].

Formally, we can start from defining the short-rate

$$r_t = X_t^1 + X_t^2 \tag{2}$$

The short-rate is therefore a direct function of only two of the three factors of the vector X_t, but the short-rate dynamics will be dependent on the three state variables.

The risk premium vector is of the essentially affine form:

$$\Gamma_t = \gamma^0 + \gamma^1 X_t \tag{3}$$

For now we choose the system of differential equations that rules the motion of X_t to be of the generic form in Equation (4).

$$dX_t = K^{\mathbb{P}}[\theta^{\mathbb{P}} - X_t]dt + \Sigma dW_t^{\mathbb{P}} \tag{4}$$

Just as in the case of the canonical affine models, the change from the equations (4) to the ones written in terms of the risk-neutral measure $\mathbb{Q}$ is done using Equation (5).

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t dt \tag{5}$$

This choice of generic risk premium enables us to have any affine form for the dynamics of the state variables in the $\mathbb{Q}$ measure. This dynamics needs to be of such a form that, using the framework of Duffie and Kan [3], we have rates as a linear function of state variables where the loadings are the ones of Nelson-Siegel [4]. Christensen et al. [1] offer proof that the dynamics of the state variables that satisfy this condition is given by Equation (6).

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \left[\begin{pmatrix} \theta_1^{\mathbb{Q}} \\ \theta_2^{\mathbb{Q}} \\ \theta_3^{\mathbb{Q}} \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt + \Sigma \begin{pmatrix} dW_t^{1,\mathbb{Q}} \\ dW_t^{2,\mathbb{Q}} \\ dW_t^{3,\mathbb{Q}} \end{pmatrix} \tag{6}$$

With this specification we get zero coupon yields between time t and T - $y(t,T)$ - expressed in the desired way – Equation (7).

$$y(t,T) = X_t^1 + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} X_t^2 + \left[\frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)} \right] X_t^3 - \frac{C(t,T)}{T-t} \tag{7}$$

Therefore, in the AFNS model, we recover almost exactly the Nelson and Siegel equation. There is a new term – the intercept of the equation – $C(t,T)$ that guarantees the non-arbitrage condition. This new term is only a function of λ – now identified as the parameter that controls the velocity of mean reversion for the slope and curvature factors – and $(T-t)$. Its functional form is given in Equation (8).

$$-\frac{C(t,T)}{T-t} = -\frac{1}{2}\frac{1}{T-t}\sum_{j=1}^{3}\int_t^T (\Sigma^T B(s,T) B(s,T)^T \Sigma)_{j,j} ds \tag{8}$$

where the vector B contains the loadings multiplied by $-(T-t)$.

2.1.1 The Independent Factor AFNS Model

Our ultimate objective for using the model is to make forecasts of interest rates. In [1] the authors compare the out-of-sample forecast performance of two nested AFNS models:

1. The Correlated-Factor AFNS – where there are no restrictions imposed on the matrix Σ[1];
2. The independent Factor AFNS, where the matrix Σ is diagonal.

The correlated-factor AFNS model, having a more complex dynamics, can replicate better the shape of the term structure and therefore have better in-sample fit. Nonetheless, complex formulations can exhibit in-sample overfitting, which is something found in [1] using six different points of the treasury's curve and two forecast periods (six and twelve months). Of the twelve combinations, the authors find better accuracy for the independent-factor AFNS in ten cases[2].

Given the results of [1] and a somewhat simpler mathematical formulations for the Independent Factor AFNS, we only consider this specific model. Therefore the matrix Σ has the specification given in Equation (9).

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{9}$$

Having defined the matrix Σ we can solve Equation (8), which leads us to Equation (10).

[1] Which means that Σ is lower (or upper) triangular. More parameters and the model would be overidentified.
[2] The exception is the shortest point of the curve (three months) for both six and twelve month's forecasts.

$$\frac{C(t,T)}{T-t} = \frac{\sigma_1 (T-t)^2}{6} + \sigma_2 f(\lambda,t,T) + \sigma_3 g(\lambda,t,T) \tag{10}$$

Where f and g are defined below.

$$f(\lambda,t,T) = \frac{1}{2\lambda^2} - \frac{1-e^{-\lambda(T-t)}}{\lambda^3(T-t)} + \frac{1-e^{-2\lambda(T-t)}}{4\lambda^3(T-t)}$$

$$g(\lambda,t,T) = \frac{1}{2\lambda^2} + \frac{e^{-\lambda(T-t)}}{\lambda^2} - \frac{e^{-2\lambda(T-t)}(T-t)}{4\lambda} - \frac{3e^{-2\lambda(T-t)}}{4\lambda^2}$$
$$- \frac{2\left(1-e^{-\lambda(T-t)}\right)}{\lambda^3(T-t)} + \frac{5\left(1-e^{-2\lambda(T-t)}\right)}{8\lambda^3(T-t)}$$

In order to obtain the term premium, we first need to forecast the interest rates for period T given the information available at time t. For this we need to forecast the state variables. Applying Ito's lemma to the variable $Y_t = e^{k^{\mathbb{P}}t}X_t$, one would get Equation (11).

$$X_T = e^{-k^{\mathbb{P}}(T-t)}X_t + \Theta^{\mathbb{P}}\left(I - e^{-k^{\mathbb{P}}(T-t)}\right) + \int_t^T e^{-k^{\mathbb{P}}(T-s)}\Sigma dW_s^{\mathbb{P}} \tag{11}$$

From Equation (11) we obtain the expected value and variance under the physical measure $\mathbb{P}$.

$$E^{\mathbb{P}}[X_T|\Im_t] = \left(I - e^{-k^{\mathbb{P}}(T-t)}\right)\Theta^{\mathbb{P}} + e^{-k^{\mathbb{P}}(T-t)}X_t \tag{12}$$

$$\begin{aligned} Q \equiv V^{\mathbb{P}}[X_T|\Im_t] &= \int_t^T e^{-k^{\mathbb{P}}(T-s)}\Sigma\Sigma^T e^{-\left(k^{\mathbb{P}}\right)^T(T-s)}dW_s^{\mathbb{P}} \\ &= \frac{\Sigma^2}{2k^{\mathbb{P}}}\left[\mathbb{I} - e^{-2k^{\mathbb{P}}(T-t)}\right] \end{aligned} \tag{13}$$

It is now necessary to estimate the parameters of the model so that we can obtain the term premium.

2.2 Model Estimation

To obtain the parameters of the model we use the Kalman filter [6] for maximum likelihood estimation.

We consider one zero-coupon interest rate curve composed of n tenors. Observations are made at N different points in time. At each time t_i we have the vector of rates $y_{t_i} \in \mathbb{R}^n$. The rates dependence on the vector of state variables X_{t_i} at time t_i is given by Equation (7), but now we include a white-noise term v_{t_i} independent of the other variables. The addition of this term means that the observation of rates are made with an error.

$$y_{t_i} = \bar{C} + \bar{L}X_{t_i} + \ v_{t_i} \tag{14}$$

where $\bar{C}$ is a vector that contains in each element $C(0, t_i)$, $\bar{L}$ is the vector of factor loadings and $v_{t_i} \sim N(0, R)$, where R is a diagonal matrix.

Because X and ν are assumed independent, and both have normal distributions[3], the distribution of y_{t_i} given $y_{t_{i-1}}$ is also Gaussian. Under the assumption that the observations are independent of each other we can write the likelihood of the data given the parameters Θ and previous realizations of the rates as in Equation (15).

$$f\left(y_{t_1}, \dots, y_{t_n}; \Theta\right) = \prod_{i=1}^{N} f\left(y_{t_{i+1}} \middle| y_{t_i}; \Theta\right) \tag{15}$$

Using logarithms and because y has normal distribution, Equation (16) gives us the function to maximize.

[3] From equation (11): $f\left(X_{t_{i+1}} \middle| X_{t_i}\right) \sim N\left(e^{-k^{\mathbb{P}}(T-t)} X_t + \Theta^{\mathbb{P}}\left(I - e^{-k^{\mathbb{P}}(T-t)}\right), Q\right)$

$$l(\Theta) = \sum_{i=1}^{N} ln\left(f\left(y_{t_i} \middle| y_{t_{i-1}}; \Theta\right)\right)$$

$$= -\frac{nNln(2\pi)}{2} - \frac{1}{2}\sum_{i=1}^{N} ln\left(\left|V^{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]\right|\right) \tag{16}$$

$$+ \left[y_{t_i} - E_{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]\right]^T V^{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]^{-1} \left[y_{t_i} - E_{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]\right]$$

The Kalman filter algorithm gives us a way to obtain the expected values and variances present in Equation (16) taking into account the uncertainty in the state variables, the uncertainty in measurements, and also the prediction error $\xi_{t_i} = y_{t_i} - E_{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]$. This can done by writing the expected value of the state variables at time t_i with information available at that time as a linear combination of the prediction for the state variable made with information available at the previous time t_{i-1} and the error of that prediction ξ_{t_i}.

$$E_{\mathbb{P}}\left[X_{t_i} \middle| \Im_{t_i}\right] = E_{\mathbb{P}}\left[X_{t_i} \middle| \Im_{t_{i-1}}\right] + K_{t_i}\xi_{t_i} \tag{17}$$

The matrix K_{t_i} is the Kalman Gain Matrix. It is obtained considering that it is a minimum variance estimator: $\frac{dtr\left(V_{\mathbb{P}}\left[X_{t_i} \middle| \Im_{t_i}\right]\right)}{dK_{t_i}} = 0$. The result is given in Equation (18).

$$K_{t_i} = V_{\mathbb{P}}\left[X_{t_i} \middle| \Im_{t_{i-1}}\right]\bar{L}^T V_{\mathbb{P}}\left[y_{t_i} \middle| \Im_{t_{i-1}}\right]^{-1} \tag{18}$$

Using $\bar{A} \equiv \left(I - e^{-k^{\mathbb{P}}(T-t)}\right)\theta^{\mathbb{P}}$ and $\bar{B} \equiv e^{-k^{\mathbb{P}}(T-t)}$ (which simplifies Equation (12)), the **Kalman Filter Algorithm** can be summarized as:

1. We start by initializing the state vector

$$E_{\mathbb{P}}\left[X_{t_1}\middle|\mathfrak{I}_{t_0}\right] = \theta^{\mathbb{P}}$$

$$V_{\mathbb{P}}\left[X_{t_1}\middle|\mathfrak{I}_{t_0}\right] = Q$$

2. At each time t_i we use the prediction for the state variables and construct the expected value and variance for the yields – needed as inputs for the objective function in (16).

$$E_{\mathbb{P}}\left[y_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right] = \bar{C} + \bar{L}E_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right]$$

$$V^{\mathbb{P}}\left[y_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right] = \bar{L}V_{\mathbb{P}}\left[y_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right]\bar{L}^T + R$$

$$\xi_{t_i} = y_{t_i} - E_{\mathbb{P}}\left[y_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right]$$

3. The expected value and variance of the state variables are updated given the error prediction and the Kalman Gain Matrix.

$$E_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_i}\right] = E_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right] + K_{t_i}\xi_{t_i}$$

$$V_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_i}\right] = \left(\mathbb{I} - K_{t_i}\bar{L}\right)V_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_{i-1}}\right]$$

4. Forecasts of the state variables needed for the next iteration (in step 2) are made using the updated expected values.

$$E_{\mathbb{P}}\left[X_{t_{i+1}}\middle|\mathfrak{I}_{t_i}\right] = \bar{A} + \bar{B}E_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_i}\right]$$

$$V^{\mathbb{P}}\left[X_{t_{i+1}}\middle|\mathfrak{I}_{t_i}\right] = \bar{B}V_{\mathbb{P}}\left[X_{t_i}\middle|\mathfrak{I}_{t_i}\right]\bar{B}^T + Q$$

5. Compute the likelihood (16).

3 Term Premium Evolution and Determinants

Going all the way back to Equation (1), we can see that we now have the complete recipe to obtain the term premium – the equation to obtain it, the model to forecast the interest rates in the physical measure, and the estimation method. We only lack one ingredient: interest rates.

With an historical data set of interest rates we can construct the term premium by estimating the model considering all the data and then forecasting interest rates. This means that for all the time points considered we forecast rates with parameters estimated using the full series. Therefore, the resultant term premium won't be the one that someone would obtain using only the data available at that time. In this way every time that we re-estimate the model using new data a new series will result, but with this choice we use all information available to infer the parameters of the model.

We start with a series of daily yields-to-maturity for every available bond. If more than three bonds are available we bootstrap the zero-rates. Using the Nelson-Siegel [4] function we obtain rates for the tenors in vector τ.

$$\tau = \left[\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1,2,3,5,7,10,15,20,25,30\right] \tag{19}$$

The model is then estimated using monthly data constructed from the daily average of τ. The last date of the sample will be in any case 30-Jun-2017. For 100 random initializations of the state vectors we get 100 parameters vectors Θ. We then choose the Θ that had the biggest likelihood in the 100 iterations.

3.1 Term Premium Evolution

For Germany the available data starts on 03-jan-1989, whereas for Portugal we only have data starting on 07-feb-1994. The restriction of four available bonds leads to a starting date of 15-mar-1996 for Portugal.

The data for all tenors in τ for Germany and Portugal is shown in Figure 1 and Figure 2.

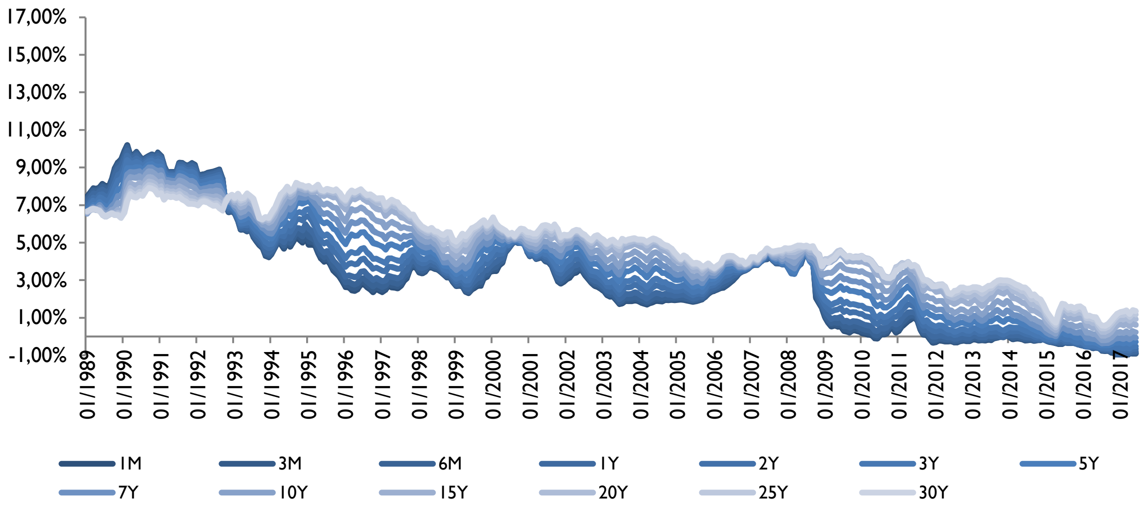

Figure 1 – Zero-coupon sovereign rates for Germany from jan-1989 to jun-2017

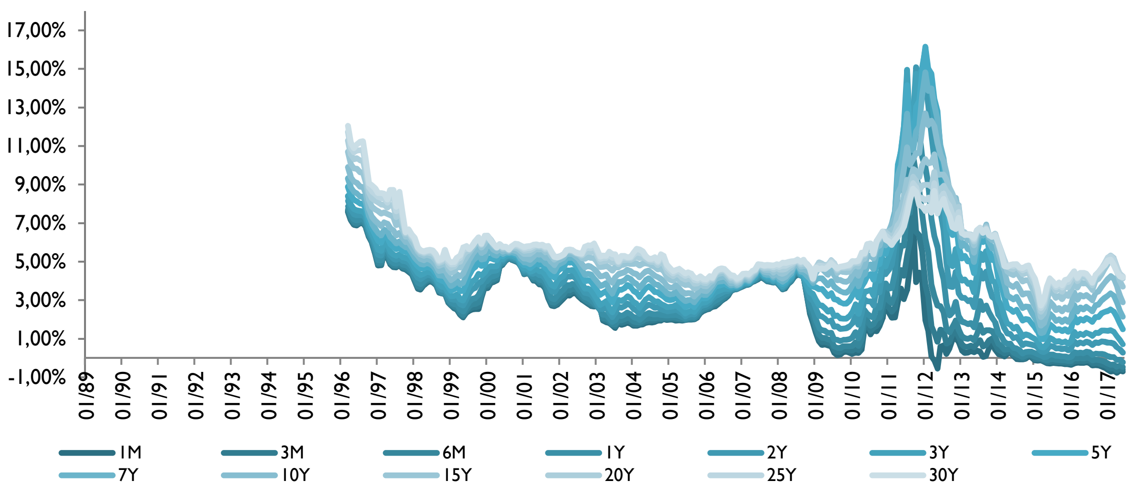

Figure 2 – Zero-coupon sovereign rates for Portugal from mar-1996 to jun-2016

We consider two different samples to estimate the model: 'Full Sample' and 'Small Sample'. The full sample uses all the values available, and small sample stars at jan-00 for both countries. In this way we have in the small sample only observations for euro denominated bonds. We can also test if choosing a different sample period results in significant changes for the estimated parameters and therefore on the term premium.

The particular term premium that we choose to compute in this study is $TP_{(1)}^{(10)}$. This gives us the yield difference between issuing a ten-year bond or issuing a one-year bond and rolling it nine times.

$$\begin{aligned} TP_{(1)}^{(10)} &= y_t^{(10)} - \frac{1}{10}\sum_{i=1}^{10} E_{\mathbb{P}}\left[y_{t+(i-1)}^{(1)} \middle| \Im_t\right] \\ &= \left(y_t^{(10)} - y_t^{(1)}\right) - \frac{1}{9}\sum_{i=2}^{10} E_{\mathbb{P}}\left[y_{t+(i-1)}^{(1)} \middle| \Im_t\right] \end{aligned} \tag{20}$$

From Equation (20) we can see that to obtain the term premium we need to forecast the value of the one-year yield from one until nine years ahead. This is done using Equation (7) and Equation (12). The results from the forecast exercise can be seen in Figure 4. To get a sense of the how reasonable the model is, the forecasts for the ten-year rate are also shown.

The full sample forecasts, containing a period of interest rates above average, leads to higher forecasts for the rates. By the end of the forecast period (which does not mean that the steady state has been reached), the samples forecasts have converged to a value of interest rates that indicates a slightly flatter interest rate curve for Germany than the observed in the full sample – 76 basis points difference between the 1Y and the 10Y rate using the full sample and 107 b.p. in the small sample, comparing with 116 b.p. in the sample average. In the case of Portugal the curve is slightly steeper: 205 b.p. in the sample average, 251 b.p. and 224 b.p. for the full and small samples, respectively.

For Portugal the forecasts made using the full sample (that include in the beginning of the period a era of high interest rates – so high that those levels were only beaten during the financial assistance program of the 2011-2014 crisis) leads to values above the sample average – 678 b.p. vs 520 b.p. for the ten year rate and 427 b.p. vs 315 b.p. for the one year rate. In the case of the small sample, the results by the end of the simulation period give us lower interest rates than the sample average – 502 b.p. for the ten year rate and 278 b.p. for the one year rate. For Germany the full sample forecasts are very similar to the full sample average: 345 b.p. vs 327 b.p. for the one year rate and 421 b.p. vs 443 b.p. for the ten year rate. In the case of the small sample the values are 213 b.p. and 320 b.p., respectively.

Making similar forecasts at all the sample points we can construct the evolution of the Term Premium.

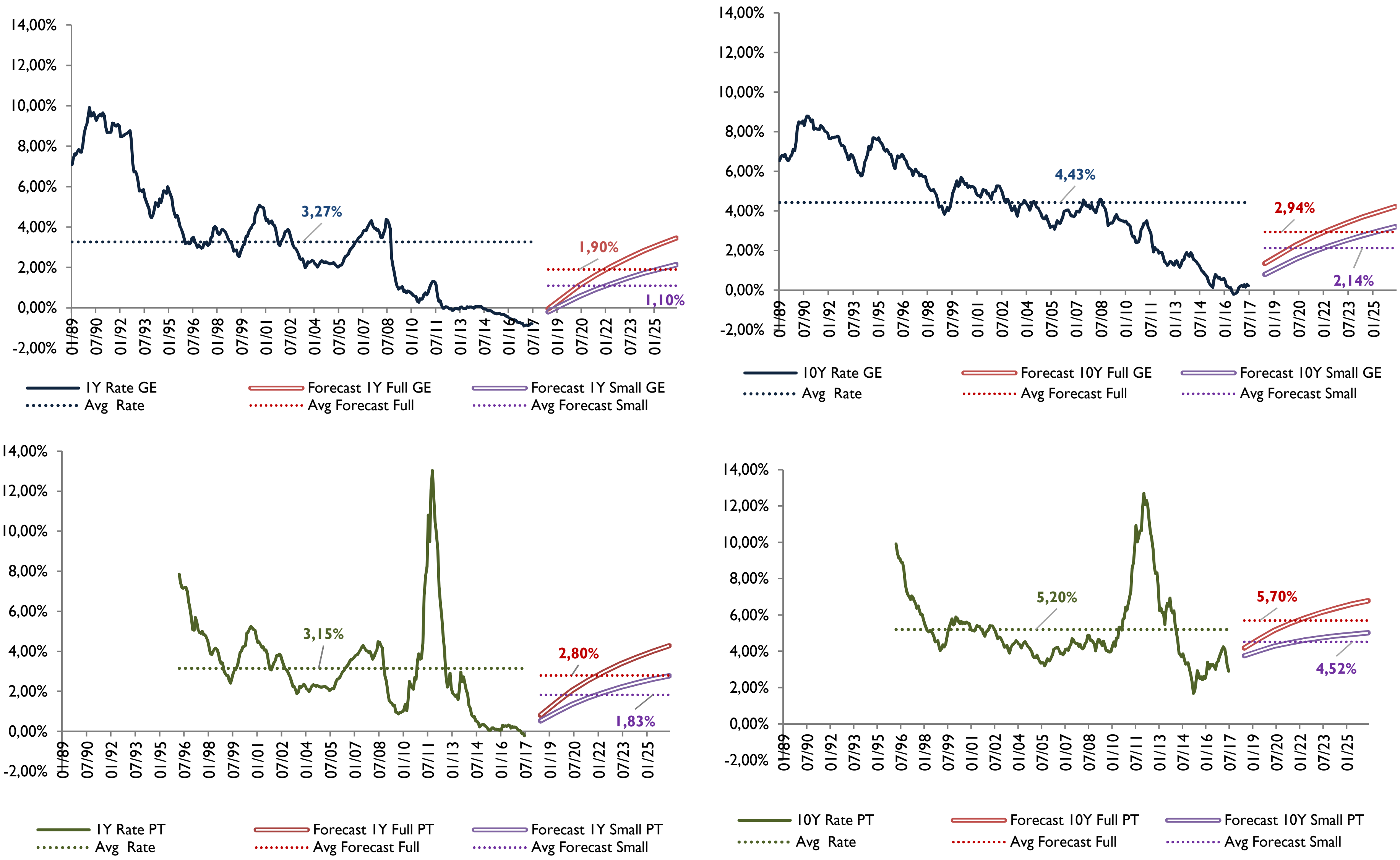

Figure 3 – Nine year forecast for the 1Y and 10Y rates for Germany and Portugal

We start the analysis with Germany, showing its term premium evolution in Figure 4

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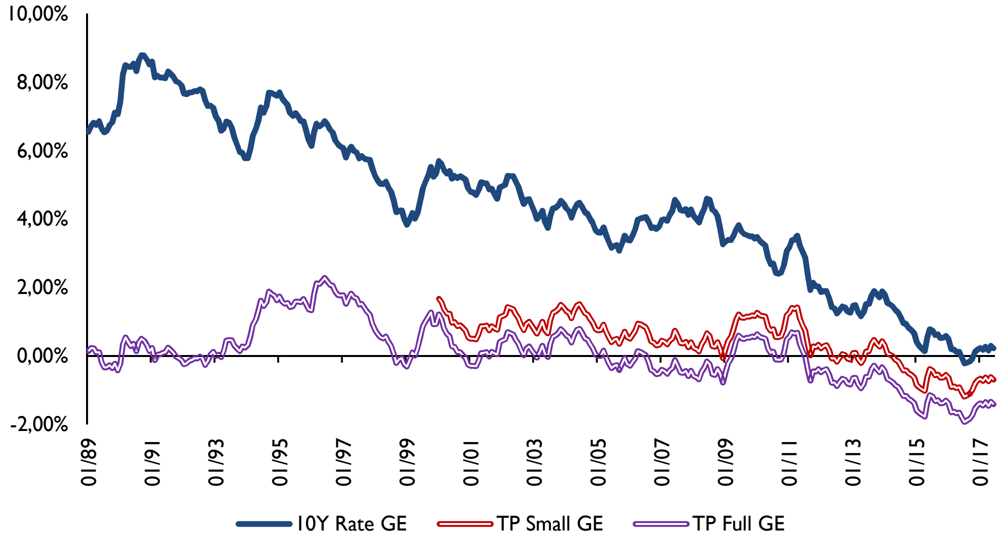

Figure 4-Term Premium evolution for Germany considering both samples

We can see that the inferred term premium is much more stable than the ten-year rate, varying in the range of -192 and 228 b.p. if we consider the full sample, and between -118 and 167 b.p. for the small sample (-21 and 879 b.p. for the ten year rate). Even though we have some absolute differences for the term premium when comparing the two samples, their behaviour is quite similar. To the naked eye it looks like one of the curves is just the other plus a shift, and in fact their difference remains in a narrow region of 14 b.p. (61-75). Therefore, the comparison between any point and all the previous would lead to the same conclusions using the full or the small sample. The premium to issue the ten year bond over the one year bond is seen as being at an all-time low at jul-2016, having recovered slightly since that date. Moreover, in both cases the model returns negative values for the term premium in the most recent period, which means that rolling-over short term debt is expected to be more expensive than issuing long term debt.

Considering Portugal, the term premium evolution follows much more closely the path of the ten year interest rate - Figure 5.

This is a consequence of having an 'outlier period' corresponding to the financial assistance program. During that period interest rates sky-rocketed but forecasts did not follow, which leads to the increase in the term premium. As in the case of Germany, the qualitative behaviour of the term premium is similar either using the full or small sample. For Portugal the models tells us for both samples that the term premium at jun-2017 had a value surpassed in history roughly half of the time, and reaching pre-crisis values. This means that the price of risk charged to increase the duration is declining, even though it is still above of what was observed before the 2008 global crisis. Nonetheless, the price of increasing duration is below the pre-crisis values, as a consequence of the low interest rates environment. Therefore, the results for Portugal are supportive of a strategy to extend duration.

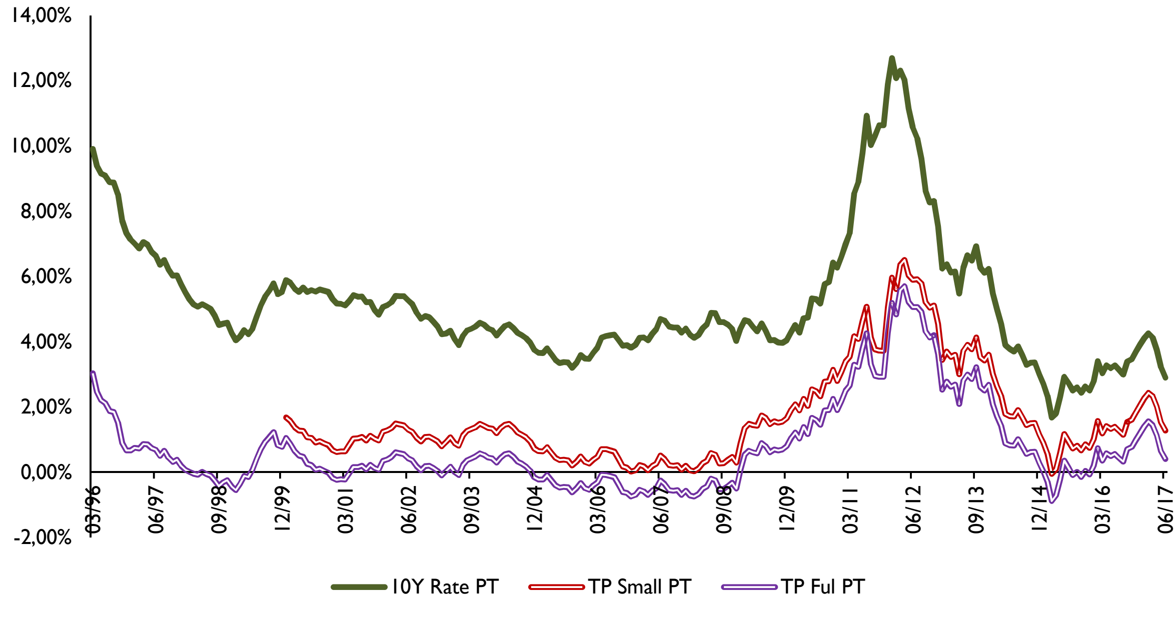

Figure 5 - Term Premium evolution for Portugal considering both samples

To complete the argument we can try to predict, for the future, the value of the term premium itself. It could be the case that the model expects the term premium to decline much further, maybe even reaching negative territory as in Germany, meaning that extending the duration should be postponed.

This is not the case when considering either of the samples. In

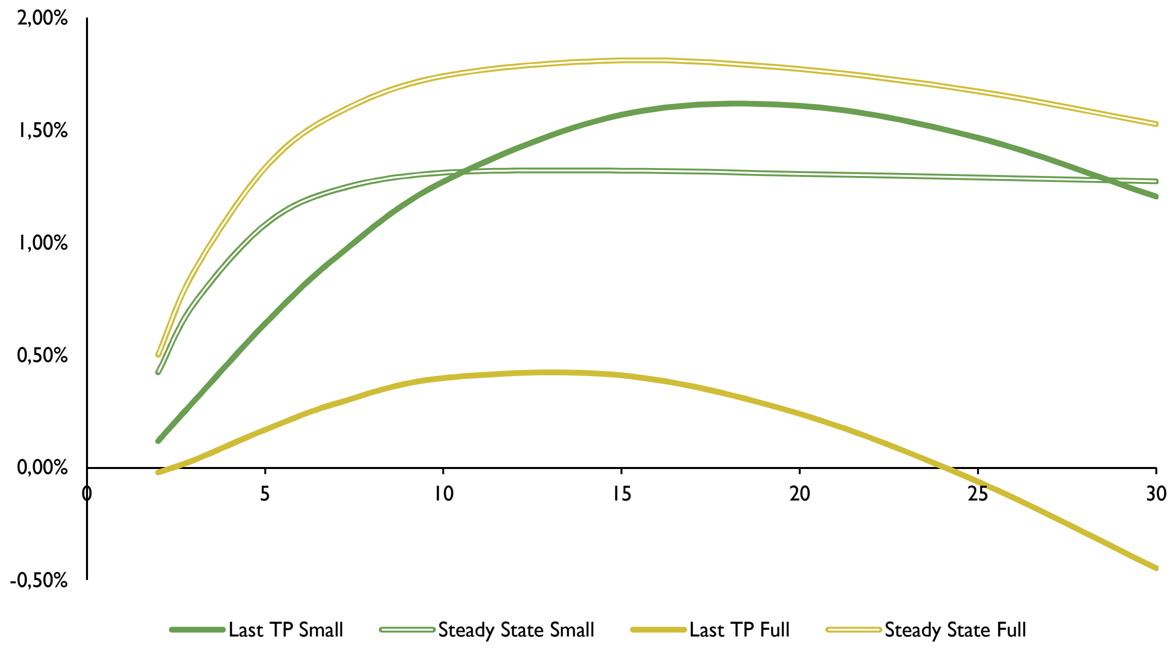

Figure 6 we have the term structure of $TP^{(m)}_{(1/12)}$. For the small sample the steady state[4] for the term structure of the term premium predicts only a slight decrease for the maturities above ten years and below thirty years, and a modest increase for maturities below ten years. This means that a discretionary choice would be to extend duration, given that the yields are low (and the observed term premium is near the steady state value). Using the full

[4] Obtained as the term premium in 50 years time

sample (that includes much more high-yield observations) the results are even more in disfavour of a postponing, given that the expectation of term premium increases.

In any case this conclusion needs to be taken with care, because if the term premium is positively correlated with the level of credit risk (as it will be shown in the next section), and knowing that the forecasts of the long term rate for Portugal are influenced by the very specific period of the financial assistance program, the steady state of the term premium given by the model is probably too high.

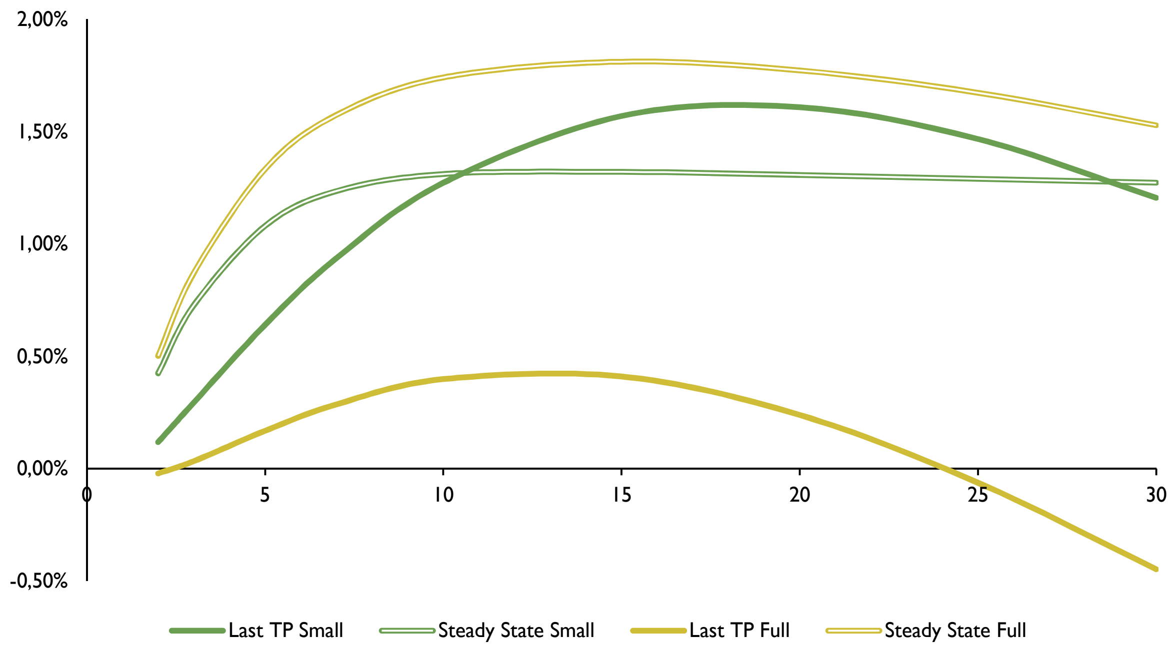

Figure 6 – The term structure of the term premium for Portugal

3.2 Term Premium Determinants

A comparison between the term premia for Portugal and Germany is made in Figure 7. We can see that their values were very similar from the start, and only started to diverge in 2009, reaching a maximum in April-2012. A convergence then started but was interrupted and reversed in the last three quarters of 2015, only to be reversed again in the second quarter of 2017.

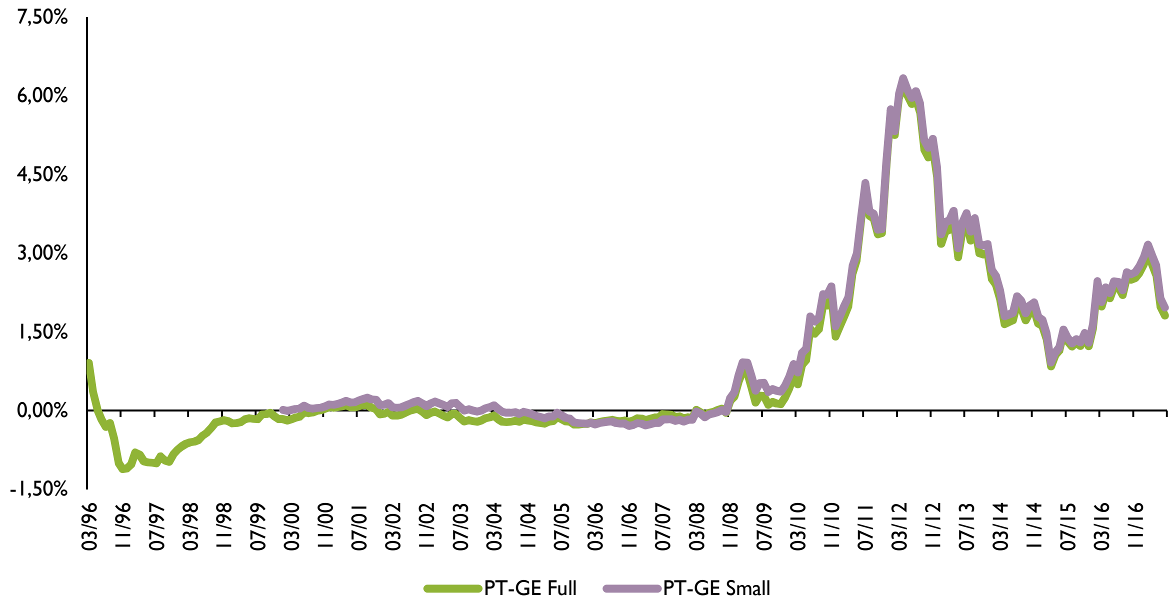

Figure 7 – Term Premium comparison between Portugal and Germany

The objective now is to understand the evolution of the term premium from variables observed in the market. This can shed light on the determinants of the term premium and give us a proxy to obtain the term premium without the need to go through the arduous process of obtaining new data and treat it, estimate the parameters and forecast interest rates.

Four variables are considered to make some simple regressions:

1. $S_1^{10} = y_t^{10} - y_t^1$ – The slope of the German curve;
2. iTraxx – Index constructed from CDS of EUR investment grade corporate bonds;
3. CISS – The ECB CISS (Composite Indicator of Market Stress) - contribution from bond market subindex;
4. $E[i]$ - Euro Inflation Swap Forward 5Y5Y.

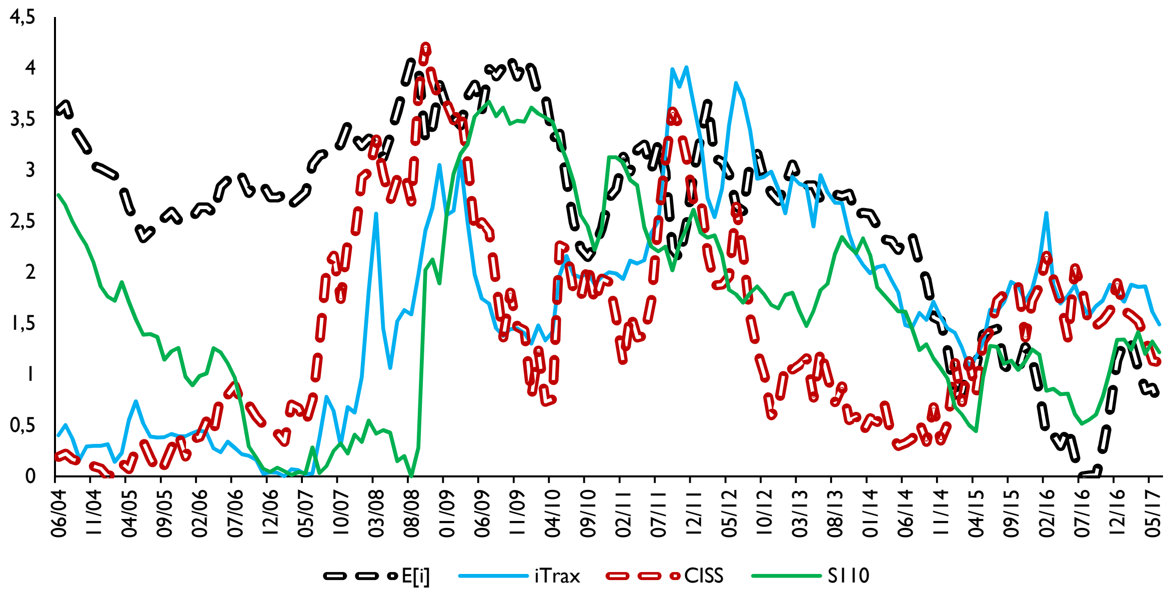

Figure 8 – The value of the variables translated by their minimum value and divided by their standard deviation (for scale purposes)

The S_1^{10} variable is used to understand if a large slope is usually correlated with increases of the term premium. It may be the case that larger slopes are correlated with larger expectations and that uncertainty on real interest rates increase with these larger expectations, which leads to a higher term premium. The iTraxx gives us an idea of the response of the term premium to measures of credit risk, and the CISS index to measures of market risk. The inflation swap is used as a proxy of inflation expectations for the period considered to estimate the term premium. This measure can be strongly correlated with the term premium if inflation uncertainty is proportional to the level of inflation.

In Table 1 we have the correlation of these variables with the German term premium estimated using the small sample and considering the period jun-2004 to jun-2017.

Correlations	*TP Germany*
S_1^{10}	54%
iTraxx	-26%
CISS	-9%
$E[i]$	83%

Table 1- Correlations between the estimated term premium and the explanatory variables for Germany

These correlations indicate that the slope of the curve can be a good explainer of the term premium, but our proxy for inflation expectations is a better one. The iTraxx and the CISS indexes have a negative correlation, which can be the consequence of a 'safe-haven' effect – increases in probabilities of default of corporates and higher market volatility leads to a reduction of the 'risk-free' rate without changes in the expectations, which means that the affected term is the term premium.

Running a linear regression with these variables for Germany (R1) we find that the coefficient for the CISS is not statistically significant. Using the remaining variables the results are given in Figure 9 and Table 2.

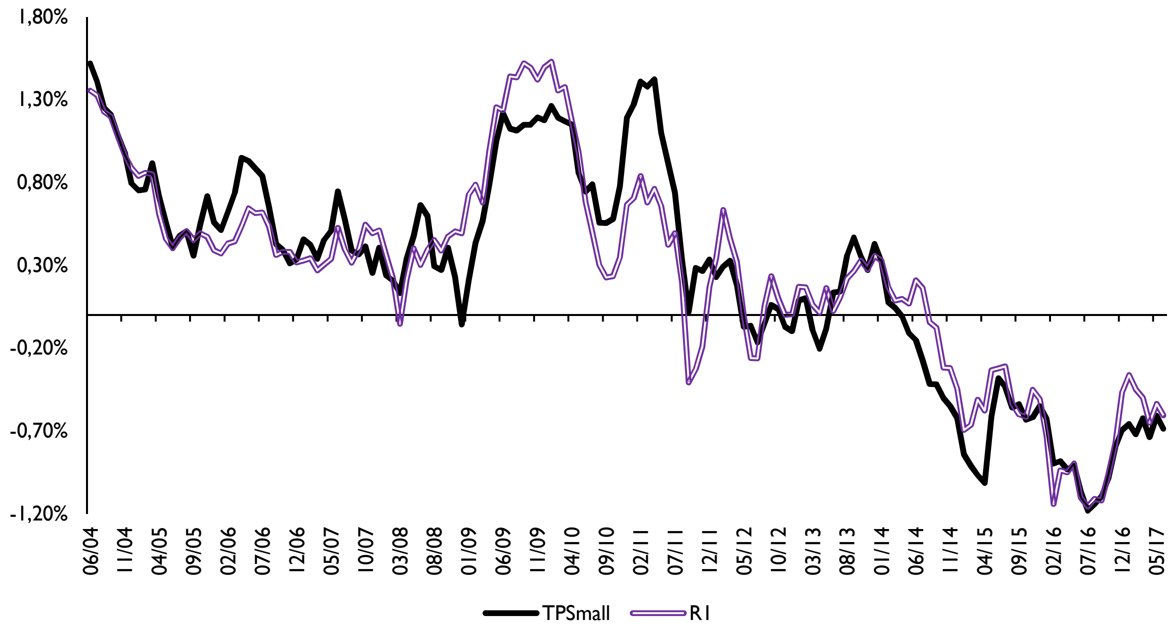

Figure 9 – Term Premium and regression equation for Germany

Variable	Coefficient Value	P-Value
Constant	-0,0226069	5E-33
$E[i]$	1.2593599	3E-43
$iTraxx$	-6,909E-05	1E-25
S_1^{10}	0,3879281	5E-24

Table 2 – Term Premium regression coefficients for Germany

The adjusted-R^2 of the regression is 86,9%. With these coefficients we need to interpret the iTraxx index as given here the 'safe-haven' effect. The negative constant is almost a necessity given the negative value of the term premium attained in the last years. Using the same interpretation for $E[i]$ and for S_1^{10} as previously, the same increase in inflation and real rate expectations results in comparatively more uncertainty in inflation than in real rates, according to the regression.

Considering now the case of Portugal, the correlation between the term premium and the explanatory variables tell us a completely different story, as seen in Table 3.

Correlations	*TP Portugal*
S_1^{10}	49%
iTraxx	76%
CISS	14%
$E[i]$	11%

Table 3 - Correlations between the estimated term premium and the explanatory variables for Portugal

Now the iTraxx has the highest correlation for the period (dominated by the post-crisis era). Moreover the correlation coefficient has a different sign from the one seen for Germany. This means that the term premium of Portugal follows much more closely the general credit risk perceived in the market than uncertainty on inflation.

The regression results (with an adjusted R^2 of 76,1%) are shown in 10 and Table 4 (without S_1^{10} because it is not statistically significant at the 5% confidence level).

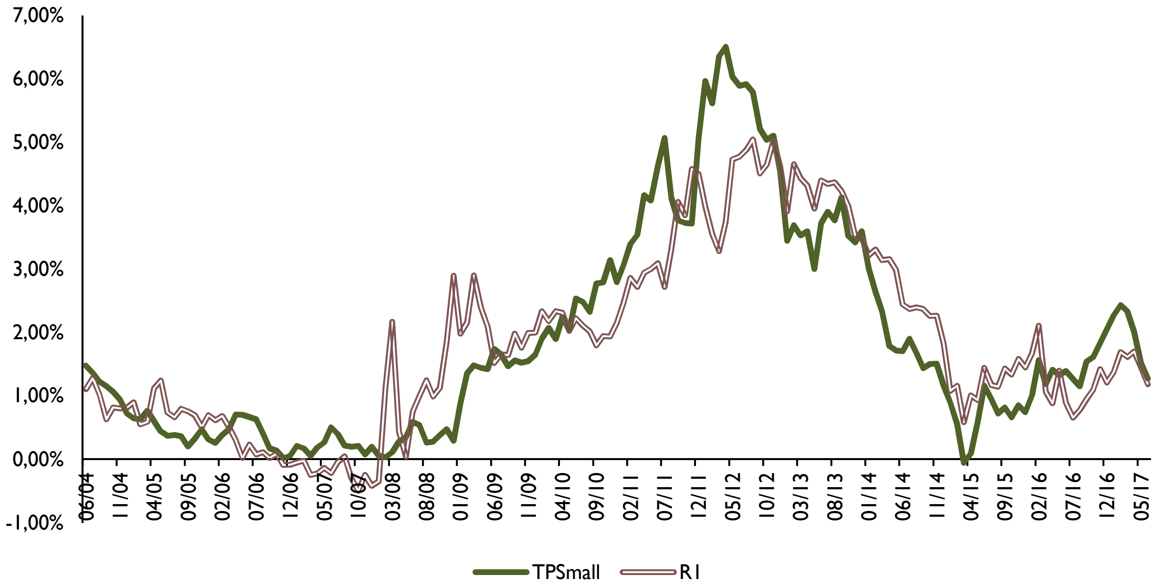

Figure 10 - Term Premium and regression equation for Portugal

Variable	Coefficient Value	P-Value
Constant	-0,0382443	1E-13
$E[i]$	1,0872224	1E-7
$iTrax$	0,0004474	4E-49
CISS	-0,275413	1E-19

Table 4 - Term Premium regression coefficients for Portugal

The dominant variable is now iTraxx and not $E[i]$. This happens because the inflation expectations have been in a declining trajectory in the period, and the estimated term premium had a large increase during the debt crisis that followed roughly the same pattern as the one for iTraxx, as seen in Figure 8.

The negative coefficient attributed to the CISS may seem strange, but it is a statistical artifact needed by the regression, because both the CISS and the iTraxx show two high-value regions, one during the global financial crisis and the other during the sovereign debt crisis, but the estimated term premium only has a significant increase in the second of these periods. Therefore the iTraxx plus CISS combination in the regression is such that it 'cancels' out the first peak, but not the second. This shows that the tem premium started to be reactive to general credit risk measures only in 2009/2010, indicating a structural break.

Provided with this information, we run a new regression from 2010 onwards. This time the CISS is also not statistically significant, and therefore the remaining variables are $E[i]$ and the iTraxx.

Variable	Coefficient Value	P-Value
Constant	-0,0586048	7E-20
$E[i]$	1,9846557	8E-12
$iTrax$	0,0003614	7E-19

Table 5 –Regression coefficients considering the period jan/10-jun/17

The adjusted R^2 improves to 79,2%, and the regression (R2) path follows more closely the estimated term premium for the considered series.

Figure 11 – Term premium and regression for Portugal in the period jan/10-jun/17

4 Concluding Remarks

In order to separate long-term interest rates into short-term rates expectations and a term premium, we used the Arbitrage-Free Nelson and Siegel Model [1]. To calibrate the model the linear Kalman filter [6] was considered using zero coupon rates constructed from yields observed in the market for all the available bonds. Two sample sizes of monthly data were used, one that included all available points and the other starting in jan-2000.

The qualitative differences between the resulting term premia were small, even though the absolute value were always higher when considering the smaller sample.

For Germany the evolution obtained indicates that we presently can ‘observe’ the term near a minimum. Adding to this that rates are also at a minimum, the decision of an issuer would be to increase duration.

For Portugal the term premium is not at a minimum, and neither is it negative, but the model expectation for future values of the term premium is not one of much lower values. This combined with the historical low interest rates, would lead a discretionary decision maker to consider a duration extension. Nonetheless, these results must be viewed with care because, as we have seen in this study, the term premium could partially be explained by the level of credit risk, and the results are dependent of the sample period that contains a very atypical period of extreme values of the credit risk.

Finally, we found a method to approximate, from market variables, the term premium obtained with the model. The German term premium shows strong correlation with inflation expectations, whereas the behaviour of Portuguese one is much more similar to those of market variables for credit risk.

Term premia are not directly observable in the market which means that different models may lead to different term premium. For this reason, the validity of the results presented in this study depends largely on the forecasting capability of the model used for projecting future short term rates, whose analysis was out of the study's scope.

The purpose of this study was to explain a methodology to estimate term premium and, as a result, to illustrate how to interpret the results to recommend a decrease or increase in duration of a portfolio or just to compare the expected costs of issuances with different maturities.

Comparing term premia obtained from alternative models and their forecasting ability would allow more definite conclusions and are left for future research.

5 References

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